Math 222A Lecture 3 Notes

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1 Well-Posedness for ODEs

1.1 Local existence and uniqueness for ODEs

Last time, we were studying a local posedness theorem for ODEs.

Theorem 1.1. *Suppose F is locally Lipschitz; i.e. the restriction to any compact set is Lipschitz. Then the ODE*

$$\begin{cases} u' = F(x,u) \\ u(0) = u_0 \end{cases}$$

has a unique local solution $u \in \mathcal{C}^1([0,T])$.

Our main tool was Banach's contraction principle.

Lemma 1.1 (Contraction principle). *Let $D \subseteq B$ be a closed subset of a Banach space, and let $N : D \to D$ be a contraction, i.e.* $\operatorname{lip}(N) < 1$. *Then N has a unique fixed point.*

This principle is useful not just in the study of ODEs but in PDEs as well. Here is a sketch of the proof.

Proof. We first prove uniqueness. Suppose $x = N(x)$ and $y = N(y)$. Then

$$\|x - y\| = \|N(x) - N(y)\| \leq \underbrace{L}_{<1} \|x - y\|.$$

This can only happen if $\|x - y\| = 0$, which implies $x = y$.

For existence, start with $x_0 \in D$. Try to improve your guess successively by setting $x_1 = N(x_0)$, $x_2 = N(x_1)$, and so on. To see that this is convergent, observe that

$$\|x_2 - x_1\| = \|N(x_1) - N(x_0)\| \leq L\|x_1 - x_0\|.$$

Iterating this gives

$$\|x_{n+1} - x_n\| \leq L^n \|x_1 - x_0\|.$$

This suggests we can think of x_n as a sort of geometric series:

$$x_n = \underbrace{x_n - x_{n-1}}_{\leq L^{n-1}} + \underbrace{x_{n-1} - x_{n-2}}_{\leq L^{n-2}} + \cdots + x_0.$$

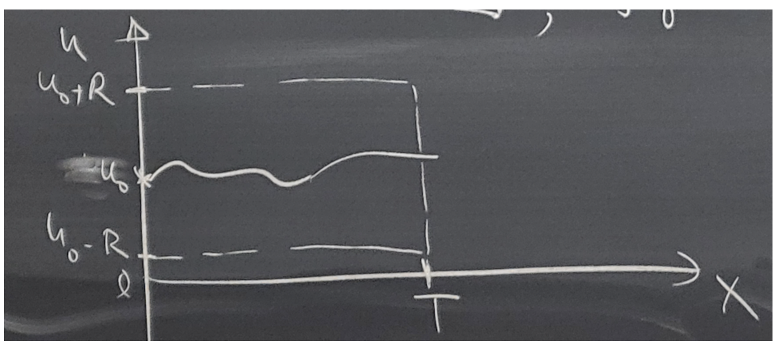
A geometric series is convergent, so the sequence x_n converges to some limit x. Since $x_{n+1} = N(x_n)$ taking the limit of both sides gives $x = N(x)$. □

This method of contraction is very useful when studying nonlinear PDEs. Now we can prove our ODE theorem:

Proof. We need N, B, and D. We obtain the map N by applying the fundamental theorem of calculus[1] to the ODE:

$$N(u)(x) = u_0 + \int_0^x F(y, u(y))\, dy.$$

Our Banach space will be $\mathcal{C}([0, T])$, where we need to figure out what is T. We want u to be locally Lipschitz, so we will define $D = \{u \in \mathcal{C}([0, T]) : \|u - u_0\|_{\mathcal{C}} \leq R\}$; we will also need to figure out what is R.



To figure out T, R, we have a few conditions:

1. We need N maps $D \to D$. For $u \in B(u_0, R)$,

$$\begin{aligned} |F(u)| &\leq |F(u_0)| + |F(u) - F(u_0)| \\ &\leq |F(u_0)| + LR \end{aligned}$$

   Suppose $R \leq 1$. Then

$$|N(u)(x) - u_0| \leq \int_0^x |F(y, u(y))|\, dy$$

[1] The idea is that the differential operator is unbounded, so you "lose" something when applying it. By contrast, when you integrate, you "gain" something.

Bound this above by the length of the integral times the size of the integrand.

$$\leq T \cdot \underbrace{(|F(u_0)| + LR)}_{C}$$

We can pick $T \ll 1$ is small enough such that

$$\leq \frac{R}{2}$$

2. N needs to be a contraction:

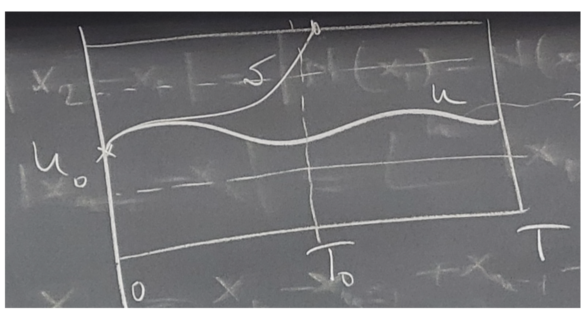
$$\begin{aligned} |N(u) - N(v)| &\leq \int_0^x |F(y, u(y)) - F(y, v(y))| \, dy \\ &\leq \int_0^x L|u(y) - v(y)| \, dy \\ &\leq T \cdot L \cdot \|u - v\|_{\mathcal{C}}. \end{aligned}$$

Picking T small enough, we get

$$\|N(u) - N(v)\| \leq \underbrace{TL}_{<1} \|u - v\|.$$

By the contraction principle, there exists a unique solution u for the integral equation in D. If u solves the integral equation, then the right hand side of the integral equation is continuous. This implies that $u \in \mathcal{C}^1$ (as integrating a continuous function gives a $\mathcal{C}^1$ function).

The other issue is that our uniqueness statement is for functions in D. For uniqueness, is there any other solution which exits $B(u_0, R)$?

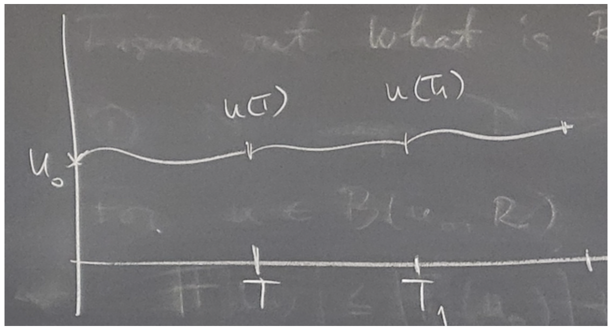


One solution is to find a T_0 small enough such that $u(T_0) \neq v(T_0)$ but $\|v - u_0\| \leq R$ in $[0, T_0]$ and apply the contraction principle in $[0, T_0]$. This gives $u = v$ in $[0, T_0]$.

Another solution is as follows. Denoting T_0 as the exit time of the ball of radius R, if $v : [0, T_0] \to B(u_0, R)$, our previous computation gives $\|v - u_0\| \leq R/2$. This is known as a **bootstrap argument**. □

1.2 Maximal solutions to ODEs

Now that we have proven existence and uniqueness of local solutions, let us move to the question of global solutions. Can we extend our local solution to global solutions?



This leads us to the idea of a maximal solution.

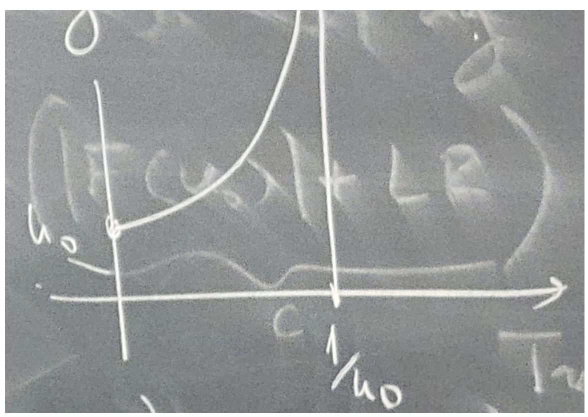
Definition 1.1. A **maximal solution** u is a solution to the differential equation that cannot be extended to a larger domain.

In general, global solutions may not exist!

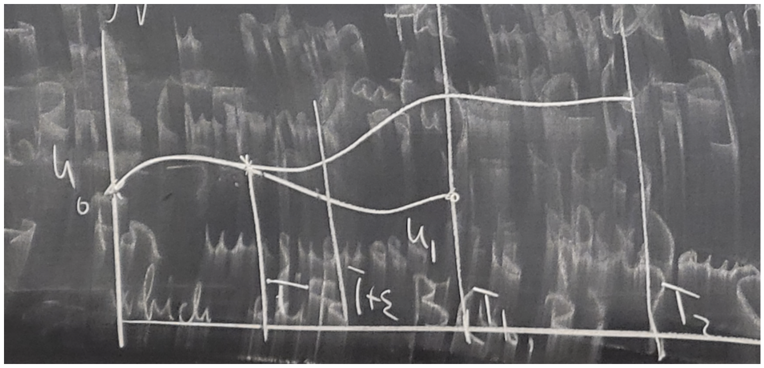
Example 1.1. Consider the equation

$$\begin{cases} u' = u^2 \\ u(0) = u_0 > 0. \end{cases}$$

By explicit computation, we can see $u(t) = \frac{1}{T-t}$, where $T = 1/u_0$.



How do we compute maximal solutions? Suppose $u_1 : [0, T_1] \to \mathbb{R}^n$ and $u_2 : [0, T_2] \to \mathbb{R}^n$ are two solutions. Can we compare them? Suppose $T_1 \leq T_2$. Then we can compare them up to time T_1.



Can this picture occur? Choose T to be maximal such that $u_1 = u_2$ in $[0, T]$. If $T < T_1$, then by local well-posedness, we must have $u_1 = u_2$ in $[T, T+\varepsilon]$. This contradicts the maximality of the choice of T, so we must have $T < T_1$. The conclusion is that as long as both solutions exists, the must be equal on the interval they share. The set of solutions is therefore ordered by inclusion, and a maximal solution exists.[2]

What can we say about maximal solutions? A maximal solution will look like $u : [0, T) \to \mathbb{R}^n$. The limit $\lim_{t\to T} u(t)$ cannot exist, or else we could solve the equation again from time T.

Proposition 1.1. *If $T < \infty$,*

$$\lim_{t\to T} |u(t)| = \infty.$$

Proof. Suppose not. Then there exists a sequence $t_n \to T$ such that $|u(t_n)| \leq M$. Start solving from t_n. We get a solution on the time interval $[t_n, t_n + T_n]$, where T_n is given by the local existence theorem. Since $|u(t_n)| \leq M$, the theorem gives $T_n = T_0$ not depending on n. If $t_n + T_0 > T$, then we get a contradiction because our solution extends beyond T. $\square$

Remark 1.1. This proposition says nothing about what will happen to global solutions.

1.3 Continuous dependence on data

Suppose $u : [0, T] \to \mathbb{R}^n$ is our reference solution with data u_0, and we vary some v with initial data v_0. We want to know if $v_0 \to u_0$, does that mean $v \to u$ in $\mathcal{C}([0, T])$?

Theorem 1.2.

(a) If $|v_0 - u_0|$ is small enough, then v exists on $[0, T]$ and satisfies $\|v - u\|_\infty \leq 1$.

[2]We do not need the axiom of choice in this case because the time intervals are totally ordered, so we can just take the union.

(b) If $v_0 \to u_0$, then $v \to u$ in $\mathcal{C}([0,T])$.

Try to track $|u-v|^2$:

$$\begin{aligned}\frac{d}{dt}|u-v|^2 &= (u-v)\cdot\frac{d}{dt}(u-v)\\ &= (u-v)(F(u)-F(v))\\ &\leq |u-v|\cdot L|u-v|\\ &= L|u-v|^2.\end{aligned}$$

We also have $|u-v|^2(0) = |u_0-v_0|^2$. Here, we have what might be called an **ordinary differential inequality** for $u-v$. If we had equality, then we would get $|u-v|^2 \leq |u_0-v_0|^2 e^{Lt}$. Otherwise, we hope to get $|u-v|^2 \leq |u_0-v_0|^2 e^{Lt}$. This step is the simplest form of what is known as Grönwall's inequality. Next time, we will discuss this inequality.